A Greedy Sparse Method Suitable for Spectral-Line Estimation

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***Abstract*—This letter presents a variant of Matching Pursuit (MP) method for compressive sensing and sparse signal reconstruction. As an extension of MP, the proposed algorithm incorporates a new backward technique to maintain or replace the previous selected atoms in the case of coherent dictionaries. Computer simulations using Fourier dictionaries are conducted to show the effectiveness of the proposed method compared to some other sparse approximation methods.**

***Index Terms*—sparse approximation, compressive sensing, exact recovery, spectral analysis, matching pursuit.**

I. Introduction

Sparse approximation with regard to a redundant dictionary has attracted much attention in recent years. Let $\mathbf{x} \in \mathbb{C}^N$ be a signal that should be recovered from the following linear measurement

$$\mathbf{\Phi x} = \mathbf{y} \tag{1}$$

where $\mathbf{\Phi} = [\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N] \in \mathbb{C}^{M \times N}$ is a dictionary with $M < N$ and $\|\boldsymbol{\phi}_n\|_2 = 1$ for $n = 1, \dots, N$.

A. State of the art

In general, the solution of the previous problem is not unique. However, when $\mathbf{x}$ is sparse, in the sense that there are a few nonzero elements in $\mathbf{x}$, it is well known that under some sufficient conditions on $\mathbf{\Phi}$, the exact recovery is possible through some non-linear convex optimization methods, such as ℓ_1 Basis Pursuit [1]. Recently, greedy algorithms received more attention due to their low computational complexity compared to ℓ_1 optimization methods. The most known greedy algorithms are: matching pursuit (MP) [4], orthogonal matching pursuit (OMP) [5], regularized OMP (ROMP) [6], compressive sampling matching pursuit (CoSaMP) [7], subspace pursuit (SP) [8] and stagewise OMP (StOMP) [9].

In the compressed sensing literature, a widely used condition on $\mathbf{\Phi}$ to ensure the exact recovery of $\mathbf{x}$ is known as Restricted Isometry Property (RIP) [2]. A matrix $\mathbf{\Phi}$ is said to satisfy the RIP condition of order K, if there exists a constant $\delta \in [0, 1)$ such that

$$(1-\delta)\|\mathbf{x}\|_2^2 \le \|\mathbf{\Phi x}\|_2^2 \le (1+\delta)\|\mathbf{x}\|_2^2$$

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for every K-sparse vector $\mathbf{x}$ (i.e. $\|\mathbf{x}\|_0 \le K$). Moreover, $\delta_K \overset{\text{def}}{=} \inf\{\delta : \text{(I-A) holds for any } K\text{-sparse } \mathbf{x}\}$ is called the isometry constant. It was shown in [10] that under the RIP condition

$$\delta_{K+1} < \frac{1}{\sqrt{K}+1}, \tag{2}$$

OMP can recover exactly the support of any K-sparse vector.

Another framework widely employed to derive conditions on $\mathbf{\Phi}$ for ensuring the exact recovery of $\mathbf{x}$ is called Mutual Coherence Property (MCP) [3]. The mutual coherence of a matrix $\mathbf{\Phi}$ is defined by

$$\mu_{\mathbf{\Phi}} = \max_{i \neq j} |\langle \boldsymbol{\phi}_i, \boldsymbol{\phi}_j \rangle|.$$

It was shown in [3] that OMP and Basis Pursuit can exactly reconstruct any K-sparse vector if the MCP condition

$$\mu_{\mathbf{\Phi}} < \frac{1}{2K-1} \tag{3}$$

is satisfied. A new result in [11] shows that exact support recovery can be guaranteed with OMP if $\mu_{\mathbf{\Phi}} < \frac{1}{K}$ and the elements of the support of $\mathbf{x}$ satisfy a decay condition, which is not easily satisfied in practice.

B. Problem statement

Motivated by the spectral analysis using sparse approximation, we are interested in the exact support recovery in the presence of dictionaries that do not satisfy either the RIP or the MCP conditions. For instance, to reconstruct a 3-sparse signal, the MCP and RIP conditions are respectively $\mu_{\mathbf{\Phi}} < \frac{1}{5}$ and $\delta_4 < 0.366$, which are strong requirements that yield to coarse frequency estimation. Nevertheless, it is known that OMP and other sparse approximation algorithms can recover exactly $\mathbf{x}$ even when the exact recovery conditions are not satisfied. However, we observe in practice that the support of $\mathbf{x}$ may not be recovered exactly from highly correlated Fourier dictionaries $\mathbf{\Phi}$ with $\mu_{\mathbf{\Phi}} \ge \frac{1}{2K-1}$, even if the signal $\mathbf{y}$ is formed by a set of atoms $\{\mathbf{a}_1, \dots, \mathbf{a}_K\} \subset \{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N\}$ that satisfy $\mu_{\mathbf{A}} < \frac{1}{2K-1}$, where $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_K]$. In this paper, we are interested in this problem.

Recently some new sparse algorithms have been proposed [12], in which continuous dictionaries are used. These algorithms deal with the off-grid frequencies problem by formulating it as a semidefinite program. Nevertheless, they are computationally expensive. This is why in [12] the authors propose to use Lasso with discrete grids as an alternative, to reduce the computational time.

Algorithm 1: Improved Matching Pursuit (IMP)

input : $\mathbf{y} \in \mathbb{C}^M$, $\mathbf{\Phi} \in \mathbb{C}^{M\times N}$ (with normalized columns)
output: An estimated support Ω.

initialization: $k = 0, \Omega_0 = \varnothing$, $\mathbf{r}_0 = \mathbf{y}$

Forward step:

while $k < K$ **do**
  $k = k + 1$
  $t_k = \underset{i\in\{1,\dots,N\}}{\arg\max} \; |\langle \mathbf{r}_{k-1}, \boldsymbol{\phi}_i\rangle|$ `// identification`
  $x_k = \langle \mathbf{r}_{k-1}, \boldsymbol{\phi}_{t_k}\rangle$
  $\mathbf{r}_k = \mathbf{r}_{k-1} - x_k \boldsymbol{\phi}_{t_k}$
end

Backward step:

$\mathbf{v}_0 = \mathbf{r}_K$
for $k = 1 : K$ **do**
  $\mathbf{q}_k = x_k \boldsymbol{\phi}_{t_k} + \mathbf{v}_{k-1}$
  $n_k = \underset{i\in\{1,\dots,N\}}{\arg\max} \; |\langle \mathbf{q}_k, \boldsymbol{\phi}_i\rangle|$ `// identification`
  $\Omega_k = \Omega_{k-1} \cup \{n_k\}$
  $\hat{\alpha}_k = \langle \mathbf{q}_k, \boldsymbol{\phi}_{n_k}\rangle$
  $\mathbf{v}_k = \mathbf{q}_k - \hat{\alpha}_k \boldsymbol{\phi}_{n_k}$
end
return $\Omega = \Omega_K, \hat{\boldsymbol{\alpha}} = \mathbf{\Phi}_\Omega^\dagger \mathbf{y}$

C. Contributions and organization of the paper

In this paper we present a new forward-backward sparse algorithm to improve the support recovery in the presence of coherent dictionaries that do not satisfy the exact recovery conditions (2) and (3). The proposed algorithm is composed of a forward step followed by a backward step. The former is nothing else but the standard MP algorithm. The latter is an estimate refinement stage in which possible wrong estimations can be rectified. So the proposed method can be considered as an improvement of the MP algorithm, hence it will be called improved MP (IMP). It is important to mention that the backward step may also be added to OMP since MP and OMP are similar forward algorithms that differ only in the residue update. We also present a sufficient condition on the atoms composing the signal $\mathbf{y}$ ensuring the exact recovery of the support of $\mathbf{x}$ by IMP.

The remainder of this paper is organized as follows. In Section II, we formulate the problem and present the proposed algorithm. Simulation results are presented in Section III. Finally, conclusions are given in Section IV.

II. Proposed Algorithm

Let $\mathbf{y}$ be a signal given by

$$\mathbf{y} = \sum_{k=1}^{K} \alpha_k \mathbf{a}_k \tag{4}$$

where $\alpha_k \in \mathbb{C}$ and $\mathbf{a}_k \in \mathbb{C}^M,\ 1 \le k \le K$. Assume that

$$\mu_{\mathbf{A}} < \frac{1}{2K-1} \tag{5}$$

with $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_K]$. We want to estimate α_k and $\mathbf{a}_k, \forall k$, from the observation $\mathbf{y}$ using a sparse approximation framework where the dictionary $\mathbf{\Phi}$ contains the components $\{\mathbf{a}_1, \dots, \mathbf{a}_K\} \subset \{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N\}$. We are particularly interested in dictionaries with mutual coherence

$$\mu_{\mathbf{\Phi}} \ge \frac{1}{2K-1},$$

otherwise the components of $\mathbf{y}$ could be recovered exactly with OMP using a dictionary satsifying the MCP or RIP conditions.

The proposed IMP method first chooses K atoms using the known stepwise forward MP method [4], and then uses a new backward technique to replace wrongly selected atoms by better ones. The process of the proposed IMP is given in Algorithm 1. IMP begins by initializing the residual with the input signal $\mathbf{r}_0 = \mathbf{y}$ and the support of $\mathbf{x}$ by the empty set $\Omega_0 = \varnothing$. Then a new atom is selected at each iteration of the forward step (MP), where the chosen atom is that with highest correlation with the dictionary . The coefficients x_k are calculated as the inner product between the current residual and the selected atom $\boldsymbol{\phi}_{t_k}$, and then the residual is updated. The backward step assumes there are wrongly selected atoms in the forward step. It begins by setting $\mathbf{v}_0 = \mathbf{r}_K$. For each iteration $k, k = 1, \dots, K$, of the backward step we put

$$\begin{aligned}\mathbf{q}_k &= x_k \boldsymbol{\phi}_{t_k} + \mathbf{v}_{k-1}\\ &= x_k \boldsymbol{\phi}_{t_k} + \sum_{i=1}^{k-1} (\alpha_i \mathbf{a}_i - \hat{\alpha}_i \boldsymbol{\phi}_{n_i}) + \sum_{p=k}^{K} (\alpha_p \mathbf{a}_p - x_p \boldsymbol{\phi}_{t_p})\\ &= \alpha_k \mathbf{a}_k + \underbrace{\sum_{i=1}^{k-1} (\alpha_i \mathbf{a}_i - \hat{\alpha}_i \boldsymbol{\phi}_{n_i}) + \sum_{p=k+1}^{K} (\alpha_p \mathbf{a}_p - x_p \boldsymbol{\phi}_{t_p})}_{\mathbf{s}_k}\\ &= \alpha_k \mathbf{a}_k + \mathbf{s}_k \end{aligned} \tag{6}$$

Therefore, $\mathbf{q}_k$ may be considered as a noisy version of $\alpha_k \mathbf{a}_k$, i.e., $\mathbf{q}_k = \alpha_k \mathbf{a}_k + \mathbf{s}_k$ where $\mathbf{s}_k$ is seen as a noise. Here again, at each iteration we select the atom the most correlated with $\mathbf{q}_k$. Then the backward step solves K sparse problems, where each one is a 1-sparse problem. This allows the backward step to have a performance that is independent of the number K of components present in $\mathbf{y}$. However, the exact support recovery becomes dependent on the level of perturbations $\|\mathbf{s}_k\|_2$. Nevertheless, $\|\mathbf{s}_k\|_2$ are guaranteed to be small since the true atoms are in the dictionary and they are sufficiently separated according to (5).

A. Discussion on the exact recovery

The following proposition provides a sufficient condition for the exact support recovery of $\mathbf{x}$.

Proposition 1: Let $\mathbf{\Phi}$ be a dictionary in $\mathbb{C}^{M\times N}$ with $\mu_{\mathbf{\Phi}} \ge \frac{1}{2K-1}$, and a signal $\mathbf{y} = \sum_{k=1}^{K} \alpha_k \mathbf{a}_k$ with $\mu_{\mathbf{A}} < \frac{1}{2K-1}$ such that $\{\mathbf{a}_1, \dots, \mathbf{a}_K\} \subset \{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N\}$. Let also Λ be the set of indices of atoms present in signal $\mathbf{y}$. If

$$\|\mathbf{s}_k\|_2 < \frac{1}{2}(1 - \delta_2)|\alpha_k|, \quad k = 1, \dots, K, \tag{7}$$

then $\Lambda \equiv \Omega$, where Ω is the support of $\mathbf{x}$ estimated in the proposed algorithm.

Proof The proof is inspired by a technique that was used in [14]. On one hand, suppose $n_k \in \Lambda$, i.e., a correct atom is selected, then

$$
\begin{aligned}
|\langle \mathbf{q}_k, \boldsymbol{\phi}_{n_k} \rangle| &= \max_{i \in \{1,\dots,N\}} |\langle \mathbf{q}_k, \boldsymbol{\phi}_i \rangle| \\
&= |\langle \mathbf{q}_k, \mathbf{a}_k \rangle| \\
&\overset{(a)}{=} |\alpha_k \langle \mathbf{a}_k, \mathbf{a}_k \rangle + \langle \mathbf{s}_k, \mathbf{a}_k \rangle| \\
&\overset{(b)}{\geq} |\alpha_k| - |\langle \mathbf{s}_k, \mathbf{a}_k \rangle| \\
&\overset{(c)}{\geq} |\alpha_k| - \|\mathbf{s}_k\|_2 \qquad (8)
\end{aligned}
$$

where (a) comes from (6), (b) is from the triangular inequality, (c) is from the definition of the inner product.

On the other hand, if a wrong atom is selected at iteration k of the forward step, i.e., $n_k \notin \Lambda$, then

$$
\begin{aligned}
|\langle \mathbf{q}_k, \boldsymbol{\phi}_{n_k} \rangle| &= |\alpha_k \langle \mathbf{a}_k, \boldsymbol{\phi}_{n_k} \rangle + \langle \mathbf{s}_k, \boldsymbol{\phi}_{n_k} \rangle| \\
&\overset{(a)}{\leq} |\alpha_k \boldsymbol{\phi}_{n_k}^{\mathsf{H}} \mathbf{a}_k| + |\langle \mathbf{s}_k, \boldsymbol{\phi}_{n_k} \rangle| \\
&\overset{(b)}{\leq} |\alpha_k| \mu_{\boldsymbol{\Phi}} + |\langle \mathbf{s}_k, \boldsymbol{\phi}_{n_k} \rangle| \\
&\overset{(c)}{\leq} |\alpha_k| \mu_{\boldsymbol{\Phi}} + \|\mathbf{s}_k\|_2 \qquad (9) \\
&\overset{(d)}{=} |\alpha_k| \delta_2 + \|\mathbf{s}_k\|_2 \qquad (10)
\end{aligned}
$$

where (a) is from the triangular inequality, (b) uses the definition of the coherence, (c) is from the definition of the inner product, (d) comes from Proposition (2.10) in [13] that shows that $\delta_2 = \mu_{\boldsymbol{\Phi}}$. To conclude, from (8) and (10), we respectively have:

$$
|\langle \mathbf{q}_k, \mathbf{a}_k \rangle| \geq |\alpha_k| - \|\mathbf{s}_k\|_2, \qquad (11)
$$

$$
|\alpha_k| \delta_2 + \|\mathbf{s}_k\|_2 \geq |\langle \mathbf{q}_k, \boldsymbol{\phi}_{n_k} \rangle|, \quad \forall \boldsymbol{\phi}_{n_k} \neq \mathbf{a}_k. \qquad (12)
$$

Yet, from hypothesis (7),

$$
|\alpha_k| - \|\mathbf{s}_k\|_2 > |\alpha_k| \delta_2 + \|\mathbf{s}_k\|_2.
$$

Now, combining this inequality with (11) and (12) leads to:

$$
|\langle \mathbf{q}_k, \mathbf{a}_k \rangle| > |\langle \mathbf{q}_k, \boldsymbol{\phi}_{n_k} \rangle|, \quad \forall \boldsymbol{\phi}_{n_k} \neq \mathbf{a}_k, \qquad (13)
$$

which shows that vectors $\mathbf{q}_k$ computed in the algorithm permit to detect the correct atoms in the dictionary, namely $\mathbf{a}_k$. ■

It is known that the MP algorithm yields similar results as OMP with Fourier dictionaries when the true number of components K is known. However OMP outperforms MP when they deal with other types of dictionaries. To get an improved OMP (IOMP) algorithm using a backward procedure as that used in IMP (Algorithm 1), we need simply to replace the forward step in Algorithm 1 by OMP algorithm [5].

Now, we present some numerical experiments evincing that condition (7) is less restrictive than the MCP condition (3). We set $M = 30$ and $K = 6$. We consider a sample of 1000 signals $\mathbf{y}$ whose components $\mathbf{a}_k$ are chosen randomly from a Fourier dictionary defined on a uniform grid of frequencies in the interval $[0, 1)$. Define the metric $\Delta_k = \frac{1}{2}(1-\delta_2)|\alpha_k| - \|\mathbf{s}_k\|_2$. Note that when $\Delta_k \geq 0$, the condition (7) is satisfied at iteration k. Figure 1 shows the value of Δ_k along the K iterations for

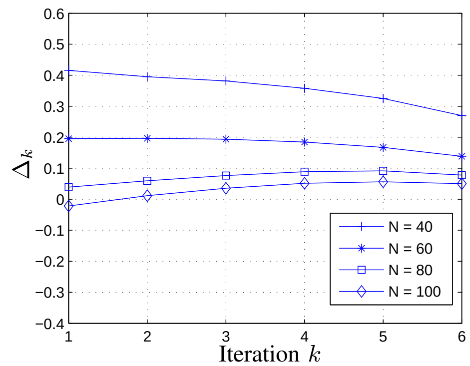

Fig. 1. Exact recovery condition at iteration k ($M = 30,\ K = 6$).

four different sizes of dictionaries: $N = 40, 60, 80,$ and 100. Therein, only at the first iteration for $N = 100$ the condition was not satisfied, which is not necessarily a drawback to estimate the correct atoms since condition (7) is sufficient and not necessary. Table below shows that the MCP condition is not satisfied for the same generated Fourier dictionaries $\boldsymbol{\Phi}$ used before, which confirms that the backward stage of IMP is an appropriate procedure to deal with coherent dictionaries.

Number of atoms	Mutual coherence $\mu_{\boldsymbol{\Phi}}$	$\frac{1}{2K-1}$
$N = 40$	0.3004	0.0909
$N = 60$	0.6369	0.0909
$N = 80$	0.7844	0.0909
$N = 100$	0.8585	0.0909

TABLE I
MUTUAL COHERENCE FOR DIFFERENT SIZES OF DICITIONARIES.

B. Computational complexity

Regarding the computational complexity, the backward step has the same cost as MP algorithm which is $O(MNK)$ where M is the size of the signal $\mathbf{y}$, N the number of atoms in the dictionary and K the sparsity level (number of components in the signal). Then the computational complexity of IMP is linear as a function of data size.

III. COMPUTER RESULTS

In order to asses the performance of the proposed algorithm, we compare it to some other sparse algorithms using computer simulations. The performance is measured by the rate of true support recovery. A recovered support is considered true if all atoms $\mathbf{a}_k, k = 1, \dots, K$, are exactly selected. We consider the harmonic spectral estimation framework.

The following algorithms are considered in our comparisons: MP [4], OMP [5], ROMP [6], CoSaMP [7], StOMP [9], SPICE [15], Lasso [16]. Since SP [8] is very similar to CoSaMP, it is not included herein. A thresholding is performed on the solutions of ROMP, StOPM, SPICE and Lasso. We use the SparseLab[1] implementation to solve the the Lasso problem. Results of MP are the same as OMP in our simulations, so we remove them for the sake of clarity in figures.

[1] https://sparselab.stanford.edu

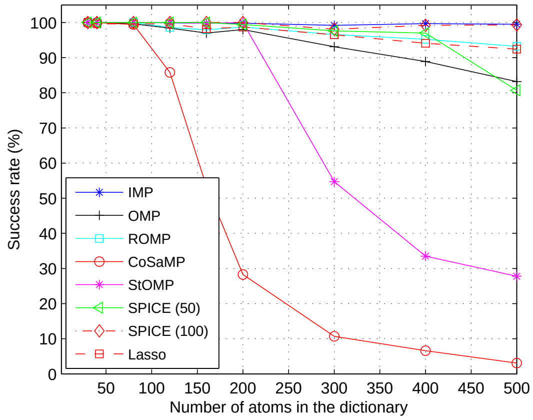

Fig. 2. Exact support recovery rate with respect to the number of atoms in the dictionary N. Number of components $K = 2$. Number of trials: 1000. Noiseless case.

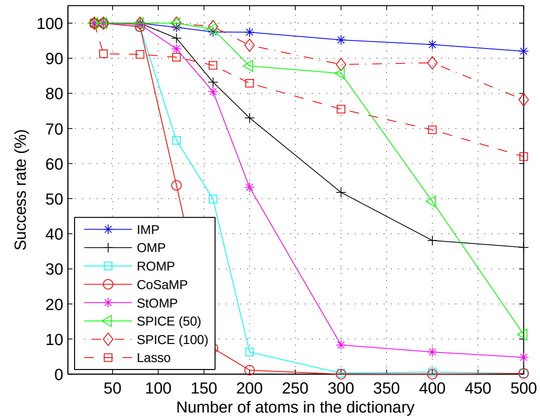

Fig. 3. Exact support recovery rate with respect to the number of atoms in the dictionary N. Number of components $K = 6$. Number of trials: 1000. Noiseless case.

The Fourier dictionary $\mathbf{\Phi}$ is defined on a uniform grid of frequencies in $[0, 1)$. The components $\{\mathbf{a}_1, \ldots, \mathbf{a}_K\}$ of $\mathbf{y}$ are chosen randomly from the dictionary so that they satisfy $\mu_{\mathbf{A}} < \frac{1}{2K-1}$. The size of generated signals $\mathbf{y}$ is 30. The real and imaginary parts of coefficients $\alpha_k, k = 1, \ldots, K$, are generated according to a uniform measure in $[0.5, 1.5]$. In all experiments, the success rate of exact support recovery is computed upon 1000 trials.

The first experiment consists of trials where noiseless signals composed of two components ($K = 2$) are considered. Figure 2 depicts the success rate with respect to the number of columns in the dictionary. SPICE (50) and SPICE (100) denote SPICE running up to 50 and 100 iterations, respectively. We see that IMP and SPICE (100) show similar results and have better success rate than the other methods.

The second experiment is similar to the first one, only the number of components changes, which is set to six ($K = 6$). Results are presented in Figure 3. We can observe that IMP outperforms the other methods for $N > 200$ atoms.

In the third experiment, we corrupt the signals of the first experiment by an additive zero-mean complex Gaussian white noise. The signal-to-noise ratio (SNR) is set to 20 dB. Results are depicted in Figure 4. We observe that IMP outperforms the other methods.

Settings of the fourth experiment are similar to the previous one except the number of components which is set to six ($K = 6$). Figure 5 presents the obtained results. Again, IMP performs better than the other methods. In summary, IMP exhibits better success rates even in the presence of noise.

IV. Conclusion

In this letter, we proposed a new greedy method called IMP for sparse reconstruction problems. IMP improves the well known stepwise forward Matching Pursuit method by adding a backward step which is based on several 1-sparse approximation problems. This method is suitable to recover sparse signals from incoherent dictionaries, in particularly Fourier dictionaries. The simulation results show that IMP outperforms OMP and some other methods.

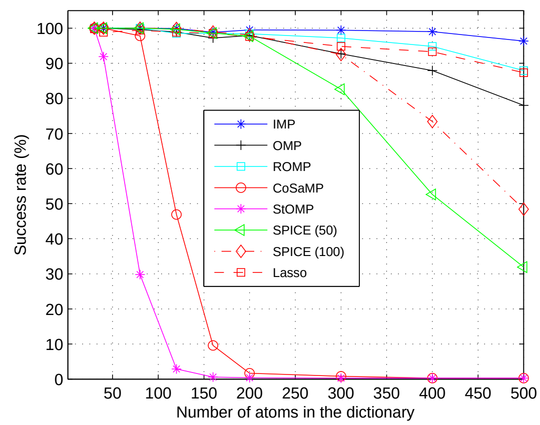

Fig. 4. Exact support recovery rate with respect to the number of atoms in the dictionary N. Number of components $K = 6$. Number of trials: 1000. SNR = 20dB.

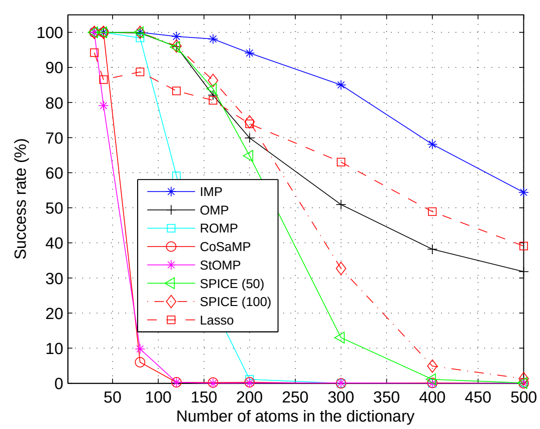

Fig. 5. Exact support recovery rate with respect to the number of atoms in the dictionary N. Number of components $K = 2$. Number of trials: 1000. SNR = 20dB.

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